International Journal of Mathematics and Computer Applications Research (IJMCAR)
ISSN(P): 2249-6955; ISSN(E): 2249-8060
Vol. 4, Issue 2, Apr 2014, 19-28

IMAGINARY ROOTS AND DYNKIN DIAGRAMS OF QUASI HYPERBOLIC KAC MOODY ALGEBRAS OF RANK 3

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ABSTRACT

Kac-Moody algebras is one of the modern fields of mathematical research which has been developing at a rapid pace in recent years mainly due to its interesting connections and applications to different fields of Mathematics and Mathematical Physics. Kac Moody algebras can be broadly classified into 3 classes: Finite, Affine and Indefinite types. The indefinite type of Kac Moody algebras can be further divided into hyperbolic and non hyperbolic types. In this paper, the properties of quasi hyperbolic Kac Moody algebras of rank 3 are studied; for the rank 3 GCM, the associated Dynkin diagrams of extended hyperbolic type are the same as that of quasi hyperbolic types. The minimal imaginary roots are explicitly computed for the rank 3 quasi hyperbolic type of Kac Moody algebras. The complete classification of Dynkin diagrams of rank 3 quasi hyperbolic type of Kac Moody algebras is obtained.

AMS MSC 2010 Code 17B67

KEYWORDS*:* Kac Moody Algebras, Generlized Cartan Matrix(GCM), Roots, Finite, Affine, Indefinite, Dynkin Diagram

1. INTRODUCTION

The theory of Kac Moody algebras was developed simultaneously and independently by V.G. Kac and R. V. Moody around 1969.While the structure of finite and affine types of Kac-Moody algebras has been well developed and understood, the structure of indefinite type of Kac Moody algebras remains to be explored completely. The root systems and the behavior of roots (real and imaginary) is better known in the case of finite, affine and hyperbolic type of Kac Moody algebras[4] , [5] etc. D. Casperson [2] studied about the strictly imaginary roots; special imaginary roots were studied by Bennett in [1]; Hechun Zhang and Kaiming Zhao in [3] defined quasi finite type of GCM's and studied the properties of imaginary roots of a class of quasi finite type GCM's. Sthanumoorthy and Uma Maheswari (1996) introduced a new class of indefinite non hyperbolic class called extended hyperbolic Kac Moody algebras and a class of roots called purely imaginary roots in [7]; The multiplicities of roots up to level 3 were also computed for specific classes of these extended hyperbolic algebras [6], [7], [8], [9].The concept of quasi hyperbolic Kac Moody algebras was introduced by Uma Maheswari [11].

In this paper, a study on the class of indefinite type of Kac Moody algebras called Quasi Hyperbolic (QH) type is done; Basic properties of QH type of Dynkin diagrams are studied; A comparative study on extended hyperbolic Kac Moody algebras and quasi hyperbolic Kac Moody algebras is done; The extended hyperbolic Kac Moody algebras form a subclass of quasi hyperbolic type of Kac Moody algebras. The minimum and maximum ranks of a Dynkin diagram of quasi hyperbolic Kac Moody algebra is 3 and 11 respectively. The complete classification of Dynkin diagrams of rank 3 quasi hyperbolic type is given.

2 PRELIMINARIES

For the basic study on Kac Moody algebras and detail proofs one can refer [4].

Definition 2.1[4]: An integer matrix A = $(a_{ij})_{i,j=1}^{n}$ is a Generalized Cartan Matrix (abbreviated as GCM) if it satisfies the following conditions:

- $a_{ii} = 2 \ \forall$ i =1,2,….,n
- $a_{ij} = 0 \Longleftrightarrow a_{ji} = 0 \ \forall$ i,j =1,2,…,n
- $a_{ij} \leq 0 \ \forall$ i,j = 1,2,…,n.

Let us denote the index set of A by N = {1,2,…,n}. A GCM A is said to decomposable if there exist two non-empty subsets I, J$\subset$N such that I $\cup$ J=N and $a_{ij}=a_{ji}=0 \ \forall$ i $\in$ I and j$\in$ J. If A is not decomposable, it is said to be indecomposable.

Definition 2.2[4]: A GCM A is called symmetrizable if DA is symmetric for some diagonal matrix D =diag $(q_1,\ldots,q_n)$, with $q_i > 0$ and q_i's are rational numbers.

Definition 2.3[4]: A realization of a matrix A = $(a_{ij})_{i,j=1}^{n}$ of rank l is a triple (H, π,π^{V}) where H is a 2n-l dimensional complex vector space, $\pi = \{\alpha_1,\ldots,\alpha_n\}$ and $\pi^{v} = \{\alpha_1^{v},\ldots,\alpha_n^{v}\}$ are linearly independent subsets of H^* and H respectively, satisfying $\alpha_j(\alpha_i^{v}) = a_{ij}$ for i,j=1,….,n. π is called the root basis. Elements of π are called simple roots.

The root lattice generated by π is $Q = \sum_{i=1}^{n} Z\alpha_i$.

The Kac-Moody algebra g (A) associated with a GCM A = $(a_{ij})_{i,j=1}^{n}$ is the Lie algebra generated by the elements e_i, f_i, i =1,…, n and H with the following defining relations:

$[h,h'] = 0$, $h,h' \in H$, $[e_i,f_j] = \delta_{ij}\alpha_i^{v}$, $[h,e_j] = \alpha_j(h)e_j$

$[h,f_j] = -\alpha_j(h)f_j, \ i,j \in N$,

$(ad\ e_i)^{1-a_{ij}}e_j = 0$

$(ad\ f_i)^{1-a_{ij}}f_j = 0 \quad \forall\ i \neq j,\ i,j \in N$

The Kac-Moody algebra g (A) has the root space decomposition

$$g(A) = \underset{\alpha\in Q}{\oplus} g_\alpha(A), \text{Where } g_\alpha(A) = \{x \in g(A)/[h,x] = \alpha(h)x,\ for\ \ all\ \ h \in H\}.$$

An element α , $\alpha \neq 0$ in Q is called a root if $g_\alpha \neq 0$. Let $Q_+ = \sum_{i=1}^{n} Z_+\alpha_i$. Q has a partial ordering "$\leq$" defined by $\alpha \leq \beta$ if $\beta - \alpha \in Q_+$, where $\alpha,\beta \in$ Q

Definition 2.4[4]: Define $r_i \in$ End (H^*) as $r_i(\alpha) = \alpha - \langle\alpha_i^{v},\alpha\rangle\alpha_i$ where $\langle\alpha_i^{v}\alpha\rangle = \alpha(\alpha_i^{v})$ and i$\in$ N. For each

i,r_i is an invertible linear transformation of H^* and r_i is called a fundamental reflection. Define the Weyl group W to be the subgroup of aut(H^*) generated by {r_i,i$\in$ N}. For any $\alpha \in Q$ and $\alpha = \sum_{i=1}^{n} k_i \alpha_i$, define support of α , written as supp α , by supp $\alpha = \{i \in N / k_i \neq 0\}.$ Let $\Delta(= \Delta(A))$ denote the set of all roots of g(A) and Δ_+ the set of all positive roots of g(A). We have $\Delta_- = \Delta_+$ and $\Delta = \Delta_+ \cup \Delta_-$

Definition 2.5[5]: To every GCM A is associated a Dynkin diagram S (A) defined as follows: S (A) has n vertices and vertices i and j are connected by max {$|a_{ij}|$,$|a_{ji}|$} number of lines if $a_{ij}.a_{ji} \leq 4$ and there is an arrow pointing towards i if $|a_{ij}|>1$. If $a_{ij}.a_{ji} > 4$, i and j are connected by a bold faced edge, equipped with the ordered pair ($|a_{ij}|$,$|a_{ji}|$) of integers.

Definition 2.6[7]: Let $\alpha \in \Delta_+^{im}.$ We say that α is purely imaginary if for any $\beta \in \Delta_+^{im}, \ \alpha + \beta \in \Delta_+^{im}.$ Similarly we say that a negative root $\gamma \in \Delta_-^{im}$ is purely imaginary if $-\gamma$ is a purely imaginary root.

$\Delta_+^{pim}(A) = \Delta_+^{pim} = \{\alpha \in \Delta_+^{im} / \alpha$ is purely imaginary} and $\Delta_-^{pim}(A) = \Delta_-^{pim} = \{\alpha \in \Delta_-^{im} / \alpha$ is purely imaginary}.

Then the set of all purely imaginary roots is $\Delta^{pim} = \Delta_+^{pim} \cup \Delta_-^{pim}.$ The symmetry of the root system means that we need only to prove the results for positive imaginary roots.

Definition 2.7[7]: We say that a GCM A satisfies the purely imaginary property if $\Delta_-^{pim}(A) = \Delta_+^{im}(A).$ we say that the Kac-Moody algebra g (A) has the purely imaginary property if A satisfies the purely imaginary property.

Remark: For all indecomposable symmetrizable GCM A = $(a_{ij})_{i,j=1}^{n}$, we have the following:

When $n \leq 4, \ \Delta_+^{im}(A) = \Delta_+^{pim}(A)$

Definition 2.8[7]: We define an indefinite non-hyperbolic, GCM A = $(a_{ij})_{i,j=1}^{n}$ to be of extended hyperbolic type if every proper, connected sub diagram of S(A) is of finite, affine or hyperbolic type.

Theorem 2.9[7]: Every indecomposable, symmetrizable extended hyperbolic Kac-Moody algebra g(A) satisfies the purely imaginary property.

Definition 2.10 [3]: $\alpha \in \Delta_+^{im}$ is called a minimal imaginary root (MI root, for short) if α is minimal in Δ_+^{im} with respect to the partial order on H^*.

3. QUASI HYPERBOLIC CLASS OF KAC MOODY ALGEBRAS

In this section we study the class of Dynkin diagrams of indefinite type called as Quasi hyperbolic type (introduced by the author in [11]) and study some of its properties.

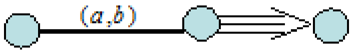
Definition 3.1: Let A = $(a_{ij})_{i,j=1}^{n}$ be an indecomposable GCM of indefinite type. We define the associated Dynkin diagram S (A) to be of Quasi Hyperbolic (QH) type if S (A) has a proper connected sub diagram of hyperbolic type with n-1 vertices. The GCM A is of QH type if S (A) is of QH type. We then say the Kac Moody algebra g(A) is of QH type.

Properties of Quasi Hyperbolic Type Dynkin Diagrams

Property 3.2: Every extended hyperbolic GCM is of quasi hyperbolic type.

Proof: Any extended hyperbolic diagram S(A) with A = $(a_{ij})_{i,j=1}^{n}$ will contain a proper sub diagram of hyperbolic type with n-1 vertices. Hence every extended hyperbolic diagram is of quasi hyperbolic type also.

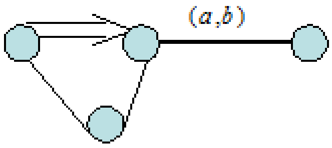
Example:



is of extended hyperbolic and also quasi hyperbolic type.

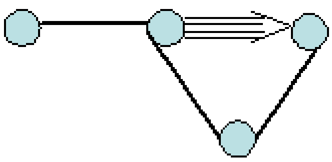
Property 3.3: A quasi hyperbolic GCM need not be of extended hyperbolic type.

The following diagram is of quasi hyperbolic type but not extended hyperbolic.



Property 3.4: There exist GCMs of indefinite type which are neither quasi hyperbolic nor extended hyperbolic.

For example, the following Dynkin diagram is neither quasi hyperbolic nor extended hyperbolic.



Theorem 3.5: Let A be a n x n GCM of quasi hyperbolic type. Then number of vertices in S (A) varies from 3 to 11.

Proof: Consider any 2 x 2 GCM $\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ where a, b are positive integers. If ab < 4, the GCM is of finite type.

If ab = 4, the GCM is of affine type. If ab > 4, the GCM is of hyperbolic type. Hence any 2 x 2 GCM cannot be of quasi hyperbolic type. Consider any Dynkin diagram of hyperbolic type with 2 vertices

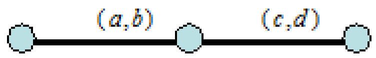
Join one more vertex to any one or two of the above vertices with one, two, three, four edges or by a bold faced edge. Then the resulting diagram will be of quasi hyperbolic type with 3 vertices. Maximum number of vertices in a Dynkin diagram of hyperbolic type is 10. Since by definition, any quasi hyperbolic diagram with n vertices should contain a sub diagram of hyperbolic type with n-1 vertices, the maximum number of vertices in a quasi hyperbolic diagram is 11.

Theorem 3.6: Let H be any connected, Dynkin diagram of hyperbolic type with n-1 vertices associated with the symmetrizable indecomposable GCM A. Let v be a new vertex joined with one or more vertices of H by 1,2,3,4 lines or by a bold faced edge. Then the resulting diagram G is of quasi hyperbolic type.

Proof: The Dynkin diagram G, constructed as in the hypothesis of the theorem will contain a proper, connected sub

diagram with n-1 vertices. By definition, G becomes a quasi hyperbolic type Dynkin diagram; This statement also gives the general construction of quasi hyperbolic diagrams;

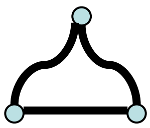
Example:



Where ab > 4, cd > 4 This Dynkin diagram is Quasi hyperbolic

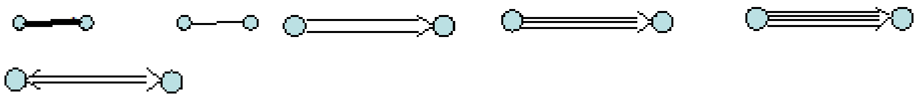
Lemma 3.7: Any Quasi hyperbolic GCM of rank 3 is also an extended hyperbolic GCM.

Proof: Let (A) be the quasi hyperbolic GCM and S(A) be the associated Dynkin diagram. Then the S(A) will be of the form



Where **the curved lines** can be joined by 1,2,3,4 edges with arrows pointing in either of directions or by a bold faced edge

Therefore, any proper connected sub diagram will be one of the following types:



Each of which is of finite, affine or hyperbolic type and hence S (A) is of extended hyperbolic type.

Properties of Imaginary Roots

Lemma 3.8: Every connected Dynkin diagram of quasi hyperbolic type with 3 vertices or 4 vertices satisfies the purely imaginary property.

Proof: Let α, β be any two positive imaginary roots; Then supp $\alpha + \beta$ will contain at least 2 vertices and hence from the characterization of Kac Moody algebras satisfying the purely imaginary property [7], it follows that $\alpha + \beta$ is also an imaginary root and hence every positive imaginary root is also purely imaginary.

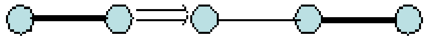
Lemma 3.9: Any completely connected quasi hyperbolic type Kac Moody algebra satisfies the purely imaginary property.

Proof*:* Since the given Dynkin diagram is completely connected, any positive imaginary root α is fully supported . Hence from the characterization for purely imaginary roots [7], α is purely imaginary.

Lemma 3.10: Any 3x3 symmetrizable quasihyperbolic satisfies the purely imaginary property.

Remark: Not all quasi hyperbolic Kac Moody algebras satisfy the purely imaginary property.

For example, in the following Dynkin diagram, $\alpha_1 + \alpha_2$ and $\alpha_4 + \alpha_5$ are positive imaginary roots.



But $\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5$ is not an imaginary root

Minimal Imaginary Roots

The minimal imaginary roots were computed for quasi finite type of indefinite Kac Moody algebras in [11]

Now let us explicitly compute the minimal imaginary (MI) roots for some classes quasi hyperbolic Kac Moody algebras.

Theorem 3.11: Let A = $\begin{pmatrix} 2 & -a & 0 \\ -b & 2 & -c \\ 0 & -d & 2 \end{pmatrix}$, ab >4, cd >4 be the GCM associated with the quasi hyperbolic Kac Moody algebra of rank 3. Then the following holds for g (A):

- For a=b,

  If a ≥ 2, $\alpha_{1+}\ \alpha_2$ is MI; If a=1, $\alpha_{1+}\ \alpha_2$ is real and $\alpha_{1+}\ 2\alpha_2$ is MI;

  If c=1, $\alpha_{2+}\ \alpha_3$ is real and $\alpha_{2+}\ 2\alpha_3$ is MI; $\alpha_{2+}\ \alpha_3$ is MI if c ≥ 2

- For a $\neq$ b,

  If a=1, $\alpha_{1+}\ \alpha_2$ is real and $\alpha_{1+}\ 2\alpha_2$ is MI;

  If a$\geq$2, $\alpha_{1+}\ \alpha_2$ is MI;

  If c=1, $\alpha_{2+}\ \alpha_3$ is real and $\alpha_{2+}\ 2\alpha_3$ is MI;

  If c ≥ 2, $\alpha_{2+}\ \alpha_3$ is MI

Proof: For A to be a symmetrizable GCM,

$$\begin{pmatrix} 2 & -a & 0 \\ -b & 2 & -c \\ 0 & -d & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b/a & 0 \\ 0 & 0 & db/ac \end{pmatrix} \begin{pmatrix} 2 & -a & 0 \\ -a & 2a/b & -ca/b \\ 0 & -ca/b & 2ca/db \end{pmatrix},$$

$(\alpha_1, \alpha_1)= 2$; $(\alpha_2,\alpha_2)=2a/b$;$(\alpha_3,\alpha_3)= 2ac/db$;$(\alpha_2,\alpha_3)= -ac/b$; $(\alpha_1,\alpha_2)=-a$; $(\alpha_1,\alpha_3)=0$

Then $(\alpha_1+\alpha_{2,}\ \alpha_1+\alpha_2) = 2+2a/b+2(-a) = 2(1+a/b-a)$

Case 1: If a=b, $(\alpha_1+\alpha_{2,}\ \alpha_1+\alpha_2) = 2(2-a)$

$$= \begin{cases} \leq 0 & \text{if } a \geq 2 \\ >0 & \text{if } a<2 \text{ i.e } a=1 \end{cases}$$

Therefore, $(\alpha_1+\alpha_{2)}$ is MI if a ≥ 2.

Suppose a=1, then b > 4, a/b < ¼; Then $(\alpha_1+\alpha_{2,}\ \alpha_1+\alpha_2) = 2/b > 0$, therefore $\alpha_1+\alpha_2$ is real.

$(2\alpha_1+\alpha_{2,}\ 2\alpha_1+\alpha_2)=4(2) + 2a/b +4(-a) > 0$; $(\alpha_1+2\alpha_{2,}\ \alpha_1+2\alpha_2) = 2+4(2a/b)+4(-a) \leq 0$

Therefore, $\alpha_1+2\alpha_2$ is a MI

Case 2: $a \neq b$

If a=1, b > 4, $(\alpha_1+\alpha_2, \alpha_1+\alpha_2) = 2(1+(a/b)-a) > 0$, Therefore, $\alpha_1+\alpha_2$ is real.

If a=2, b>2, $(\alpha_1+\alpha_2, \alpha_1+\alpha_2) = 2((a/b)-1) < 0$, therefore $\alpha_1+\alpha_2$ is MI.

If a > 2, $(\alpha_1+\alpha_2, \alpha_1+\alpha_2) = 2(1 + (a/b)-a)$.

$(\alpha_1+\alpha_2, \alpha_1+\alpha_2) < 0$ if b=1, and $\alpha_1+\alpha_2$ is MI; Also $(\alpha_1+\alpha_2, \alpha_1+\alpha_2) < 0$ if b > 1, a/b < 1 , $\alpha_1+\alpha_2$ is MI

And $(\alpha_1+\alpha_2, \alpha_1+\alpha_2) < 0$ if b >1, a > b, $\alpha_1+\alpha_2$ is MI.

Consider the case a=1, b > 4; $\alpha_1+\alpha_2$ is real; $(2\alpha_1+\alpha_2, 2\alpha_1+\alpha_2) = 4(2) + 2a/b +4(-a) > 0$;

$(\alpha_1+2\alpha_2, \alpha_1+2\alpha_2) = 2 + 4(2a/b) + 4(-a) < 0$; Therefore, $\alpha_1+2\alpha_2$ is a MI root when a=1, b > 4.

Next consider $(\alpha_2 +\alpha_3, \alpha_2+\alpha_3) = 2(a/b) + 2(ac/db)-2(ac/b)$

If a = b and c = d, $\alpha_2+\alpha_3$ is MI

If a = b, $(\alpha_2 +\alpha_3, \alpha_2+\alpha_3) = 2 + 2(c/d) - 2c > 0$, therefore $\alpha_2 +\alpha_3$ is real

If c > 2 and c < d, c/d <1, $(\alpha_2 +\alpha_3, \alpha_2+\alpha_3) < 0$, therefore $\alpha_2 +\alpha_3$ is MI

If c > 2, c > d, $(\alpha_2 +\alpha_3, \alpha_2+\alpha_3) < 0$, therefore $\alpha_2 +\alpha_3$ is MI

For c=1, $(2\alpha_2 +\alpha_3, 2\alpha_2+\alpha_3) = 4(2a/b) + 2(ac/db)+4(-ac/b) > 0$

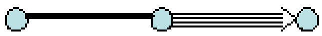
For c=1, $(\alpha_2 +2\alpha_3, \alpha_2+2\alpha_3) = 2a/b+ 4(2ac/db)+4(-ac/b) < 0$

Since c = 1, d > 4, therefore $\alpha_2+2\alpha_3$ is MI

If $a \neq b$, let a < b, $(\alpha_2 +\alpha_3, \alpha_2+\alpha_3) = (2a/b) + 4(a/db) - 2(a/b) > 0$, therefore $\alpha_2+\alpha_3$ is real.

If c = 2, $(\alpha_2 +\alpha_3, \alpha_2+\alpha_3) = (2a/b) + 4(a/db) - 4(a/b) < 0$, therefore $\alpha_2+\alpha_3$ is MI.

If c < d or c > d, again $\alpha_2+\alpha_3$ is MI Hence the theorem is proved.

Theorem 3.12: Let A = $\begin{pmatrix} 2 & -a & 0 \\ -b & 2 & -1 \\ 0 & -4 & 2 \end{pmatrix}$, ab >4 be the GCM associated with the quasi hyperbolic Kac Moody algebra of rank 3 whose Dynkin diagram  is obtained from the affine $A_2^{(2)}$ .Then the following holds for g(A) :

- If a ≥ 2, $\alpha_1+\alpha_2$ is MI; If a=1, $\alpha_1+\alpha_2$ is real and $\alpha_1+2\alpha_2$ is MI;
- $\alpha_2+\alpha_3$ is MI if c ≥ 2

Proof: Proof is direct from the computation using the non degenerate symmetric bilinear form as in the above theorem.

Theorem 3.13: Let A = $\begin{pmatrix} 2 & -a & 0 \\ -b & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$, ab >4, cd >4 be the GCM associated with the quasi hyperbolic Kac Moody algebra of rank 3, whose Dynkin diagram  is obtained from affine $A_1^{(1)}$. Then the

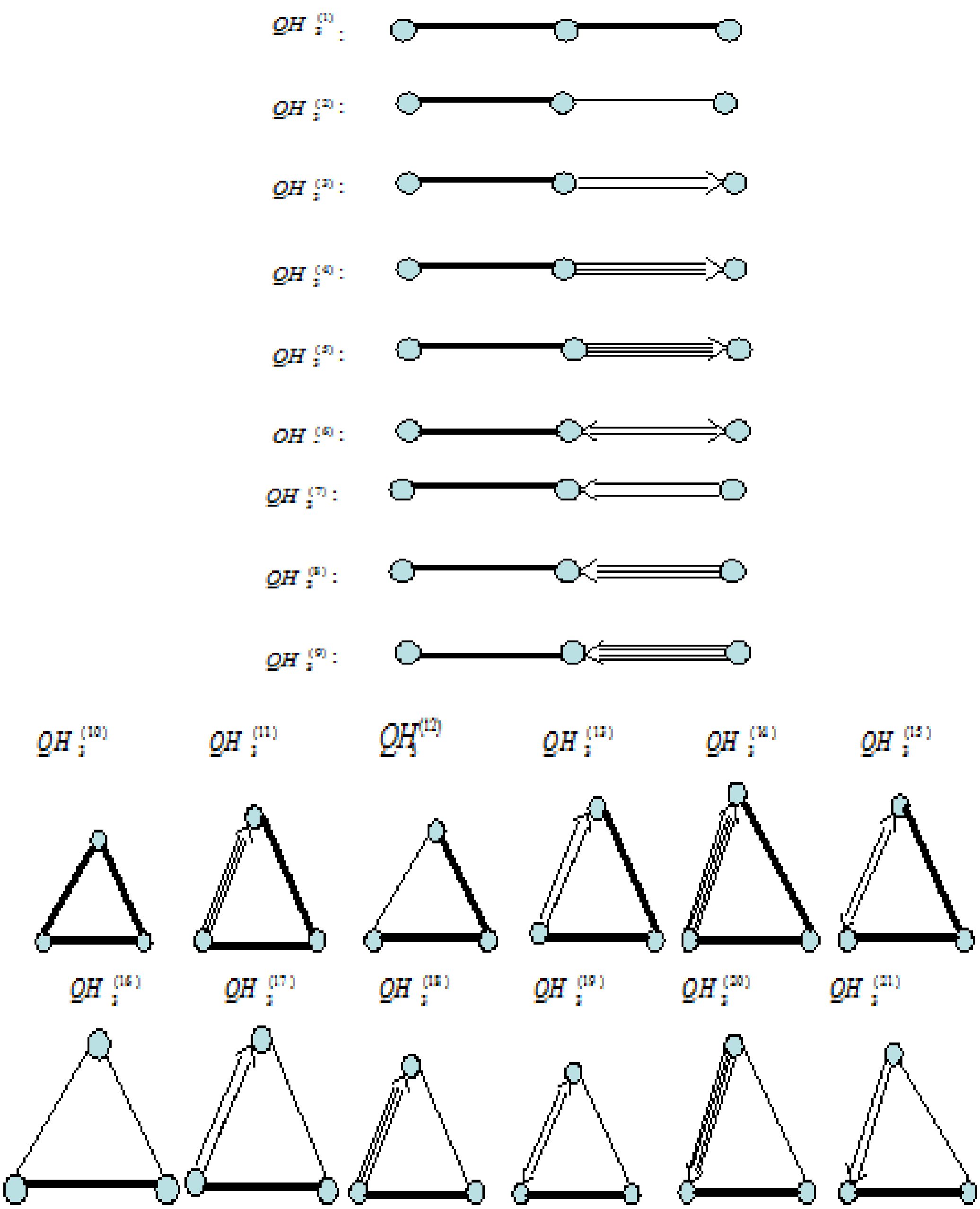
following holds for g (A):

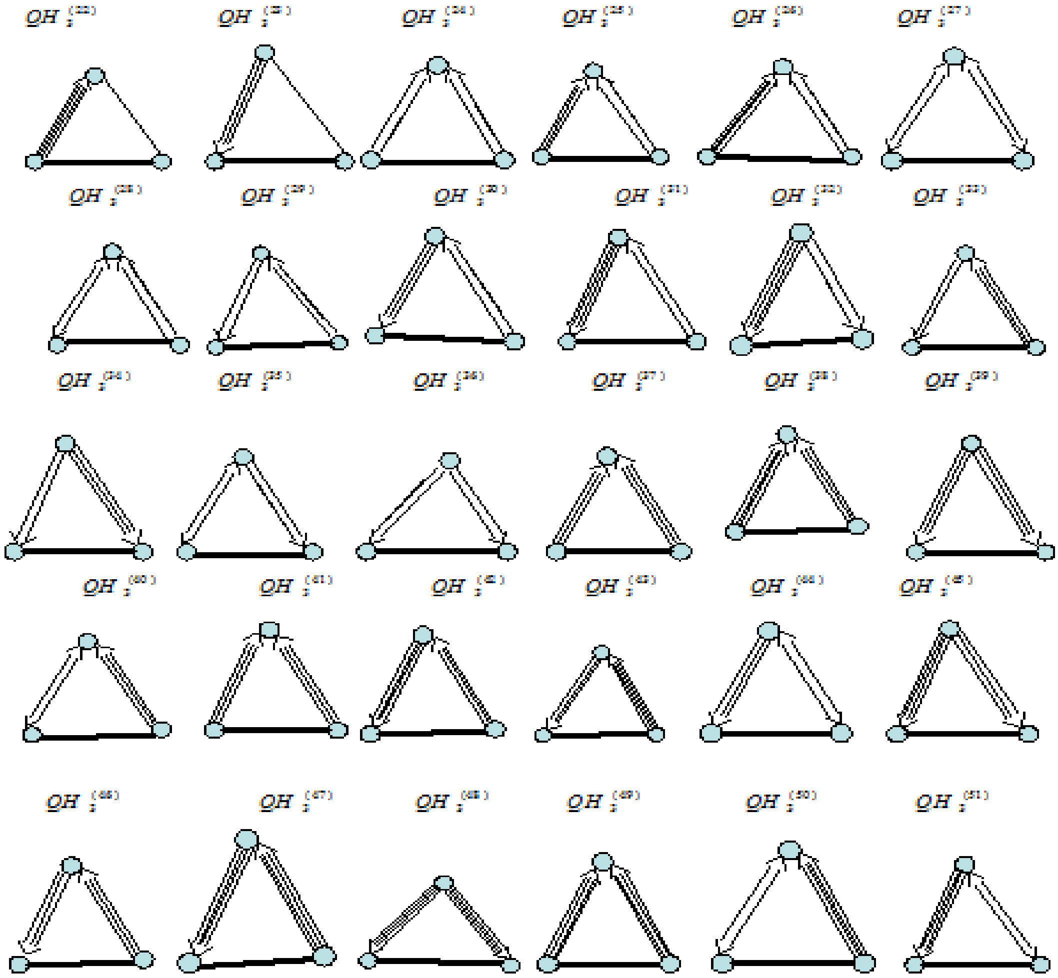
- If a ≥ 2, $\alpha_{1+}\ \alpha_2$ is MI;
- If a=1, $\alpha_{1+}\ \alpha_2$ is real and $\alpha_{1+}\ 2\alpha_2$ is MI;
- $\alpha_{2+}\ \alpha_3$ is MI if c ≥ 2

Proof: Proof is direct from the computation using the non degenerate symmetric bilinear form as in the above theorem.

The following theorem gives the complete classification of Quasi Hyperbolic type of Dynkin diagrams of rank 3.

Theorem 3.14: For a rank 3 GCM $\begin{pmatrix} 2 & -a & -m \\ -b & 2 & -c \\ -n & -d & 2 \end{pmatrix}$ of Quasi Hyperbolic type of Kac Moody algebra, the complete classification of Dynkin diagrams is given by:

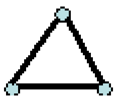


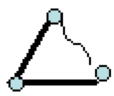


Proof: Any GCM of Quasi hyperbolic type of rank 3 must contain the bold faced edge

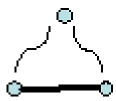
Hence atleast one of the products ab, mn or cd > 4 Therefore, S(A) has atleast one bold faced edge.

Case 1: If ab > 4, mn > 4 and cd > 4, the Dynkin diagram is

 each bold faced edge representing the pairs (a,b), (c,d) and (m,n).

Case 2: If ab > 4 and cd > 4, the Dynkin diagram contains two bold faced edges (a,b), (c,d) : 

and the product mn=1,2,3 or 4 where the curved line may be one of the following :Single edge, two edges with an arrow pointing on either of the sides or on both sides, three edges with an arrow pointing on either of the sides , four edges with an arrow pointing on either of the sides. The corresponding diagrams are then as given in the theorem with two bold faced edges.

Case 3: If exactly one of ab,cd or mn >4, then the possible Dynkin diagram is 

Hence the corresponding possible Dynkin diagrams with the single bold faced edge are as listed in the theorem.

ACKNOWLEDGEMENTS

The author sincerely acknowledges the University Grants Commission, India for sanctioning the UGC Research Award and this work is part of the research work under the UGC Research Award Scheme

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